Optimal Guidance for High-Order and Acceleration Constrained Missile

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Explicit formulas of optimal guidance for a linear, time-invariant, arbitrary-order, and acceleration constrained missile are derived. These formulas are given in terms of the missile's transfer function and acceleration constraint. Optimal, full-state feedback guidance laws are synthesized and compared to first-order approximation and the proportional navigation for minimum and nonminimum missile dynamics. Simulations on a third-order missile model show the relative gain from using the full-order guidance law vs the acceleration constraint as well as some robustness tests.

I. Introduction

THE optimal control theory has been used to derive modern/optimal guidance laws which have improved performance. The improved performance of these homing laws is achieved by a consideration of the detailed dynamics of the threat (target) and the interceptor (missile). However, it comes at the expense of increased complexity in realization (cost), sensitivity to knowledge of various parameters, etc.

An extensive study of literature on guidance laws in general, and optimal guidance laws in particular, is performed by Pastrick et al.[1] In various references,[2-6] optimal guidance laws are derived for first- and second-order missiles, respectively. In our previous paper,[7] the structure of optimal guidance laws for a linear, arbitrary high-order missile was considered. Mainly, we derived the closed-loop, general structure formulas of the guidance law. Further, we studied the behavior of the gains for minimum and nonminimum phase missiles and compared the performance of some sub-optimal approximations of the guidance laws.

The effect of the acceleration constraint, which is imposed by the structural or aerodynamic limitations, on guidance laws and performance for a first-order missile is systematically treated by Anderson.[8]

In this paper we derive an optimal/modern guidance law, on collision course, for a linear, time-invariant, arbitrary order and acceleration constrained missile. It is shown that, for a minimum phase missile, the optimal guidance law is the guidance law for an unconstrained missile with saturation on the commanded acceleration. However, for a nonminimum phase missile, this is only a suboptimal guidance law, and the optimal controller is more complicated.

In the paper comparison of the proportional navigation, first-order approximation, and full-order guidance laws is performed on a third-order minimum and nonminimum phase model of a missile. The comparison is performed on a common basis. Moreover, the robustness of these guidance laws is subject to an analysis, namely, the sensitivity to uncertainty/variation in parameters, radome refraction slope, and acceleration constraint is checked for minimum and nonminimum phase airframes.

The main conclusions are that, for a minimum phase missile, the full-order guidance law does not give improved performance with respect to the first-order approximation, whereas, for a nonminimum phase missile, there are situations (combinations of poles and zeros location and acceleration constraint) when the full-order guidance law is worth consideration.

II. General Intercept Problem and Solution

In order to derive an optimal guidance law, let us consider the minimization of the following quadratic index. This performance index is equivalent to any other quadratic unconstrained index, as shown in Appendix A.

$$J = \frac{1}{2}\left[x^T(t_f)\, Gx(t_f) + \int_t^{t_f} u^T(\tau)Ru(\tau)\, \mathrm{d}\tau\right] \tag{1}$$

where $x(\cdot)$ is the state vector; $u(\cdot)$ the control vector; $G \geq 0$ and $R > 0$ weighting matrices; and t_f the time of flight. All vectors and matrices are of appropriate dimensions. Minimization of the performance index is subject to the linear differential equation constraint

$$\mathring{x} = Ax + Bu \tag{2}$$

and a constraint on the input

$$u^T(t)\, u(t) \leq U_0^2 \tag{3}$$

In Appendix B the following solution is obtained

$$u(t) = -U_0 \operatorname{Sat}[(1/U_0)\, R^{-1}B^T\Phi^T(t_f,t)Gx(t_f)] \tag{4}$$

where the saturation vector function is defined as

$$\operatorname{Sat}[x] \begin{cases} x, & \|x\| \leq 1 \\ \dfrac{x}{\|x\|}, & \|x\| > 1 \end{cases}$$

The terminal state is given implicitly by the integral equation

$$x(t_f) = \Phi(t_f,t)x(t) - \int_t^{t_f} \Phi(t_f,\tau)BU_0 \times \operatorname{Sat}\left[\frac{1}{U_0} R^{-1}B^T\Phi^T(t_f,\tau)x(t_f)\right] \mathrm{d}\tau \tag{5}$$

where

$$\mathring{\Phi}(t,t_0) = A\,\Phi(t,t_0), \qquad \Phi(t_0,t_0) = I \tag{6}$$

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This optimal solution is usually difficult to implement. It may be approximated by the following practical solution:

$$u(t) = -U_0 \operatorname{Sat}\left\{\frac{1}{U_0} R^{-1}B^T\Phi^T(t_f,t) \times G\left[I + \int_t^{t_f} \Phi(t_f,\tau) BR^{-1}B^T\Phi^T(t_f,\tau)G\,\mathrm{d}\tau\right]^{-1} \Phi(t_f,t)x(t)\right\} \tag{7}$$

This practical solution is, for some cases that will be described in the sequel, the optimal solution. For other cases it is only a suboptimal solution.

III. Optimal Guidance Law Derivation

The intercept geometry is shown in Fig. 1. Here we use the same methodology and notation as in Ref. 7. The linearized kinematics are given by the differential equations

$$\mathring{y} = v \qquad \mathring{v} = a_T - a_m \tag{8}$$

The dynamics of the nth-order missile are

$$\begin{bmatrix} \mathring{a}_m \\ \mathring{p}_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_m \\ p_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \tag{9}$$

This is a partition of the autopilot state variables where the missile's acceleration a_m is the first state variable and p_m are the rest $n-1$ state variables. The a_{11} and b_1 are scalars; a_{21}, a^T_{12}, and b_2 are $(n-1)\times 1$ vectors; a_{22} is an $(n-1)\times(n-1)$ matrix; and $u(t)$ us a scalar input. Consequently, the system of Eq. (2) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ v \\ a_m \\ p_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y \\ v \\ a_m \\ p_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} u(t) \tag{10}$$

The contribution of a deterministic target acceleration a_T is treated in Refs. 6 and 11; therefore it will not be treated here and is left for the example. Further, let us assume, that with regard to the state variables we are interested only in the minimization of the final miss, $y(t_f)$, i.e.,

$$G = \operatorname{diag}[g, 0, \ldots, 0], \qquad R = 1 \tag{11}$$

A. General Solution

Substitution of Eqs. (10) and (11) in Eqs. (4) and (5) results in

$$u(t) = U_0 \operatorname{Sat}\left[\frac{g}{U_0}\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right)\bigg|_{t_f-t} y(t_f)\right] \tag{12}$$

where $y(t_f)$ is given implicitly by

$$y(t_f) = y(t) + (t_f - t)v(t) - \left[\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{a_m(0)}\right)\bigg|_{t_f-t}\ \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{p_m(0)}\right)\bigg|_{t_f-t}\right] \times \begin{bmatrix} a_m(t) \\ p_m(t) \end{bmatrix} - \int_0^{t_f-t} \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right)\bigg|_{t_f-\tau} \times \operatorname{Sat}\left[\frac{g}{U_0}\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right)\bigg|_{t_f-\tau} y(t_f)\right]\mathrm{d}\tau \tag{13}$$

where

$a_m(s)/u(s)$ = missile transfer function
$a_m(s)/a_m(0)$ = missile acceleration response to initial conditions in the acceleration state
$a_m(s)/p_m(0)$ = missile acceleration response to initial conditions in the states p_m, $1\times(n-1)$ vector
$\mathcal{L}^{-1}$ = the inverse Laplace transform operator

The initial condition responses $a_m(s)/a_m(0)$ and $a_m(s)/p_m(0)$ are evaluated from the homogeneous equation of the missile of Eq. (9).

Note that the following notation is used in Eq. (13):

$$\frac{v}{x} = \left[\frac{v}{x_1}, \frac{v}{x_2}, \ldots, \frac{v}{x_k}\right]$$

B. Minimum Phase Missile

This section deals with a minimum phase missile. The ramp response of a minimum missile is monotonically increasing, i.e.,

$$\mathcal{L}^{-1}\left(\frac{1}{s}\frac{a_m(s)}{u(s)}\right)\bigg|_t > 0$$

(This section applies, as well, for the more general class of missiles with monotonically nondecreasing ramp response.) Figure 2 describes the behavior of the saturation function in Eqs. (12) and (13). From Fig. 2 one can see that for $t_1 < t < t_f$ no saturation occurs and Eqs. (12) and (13) can be solved explicitly. At $t = t_1$ saturation occurs simultaneously in Eqs.

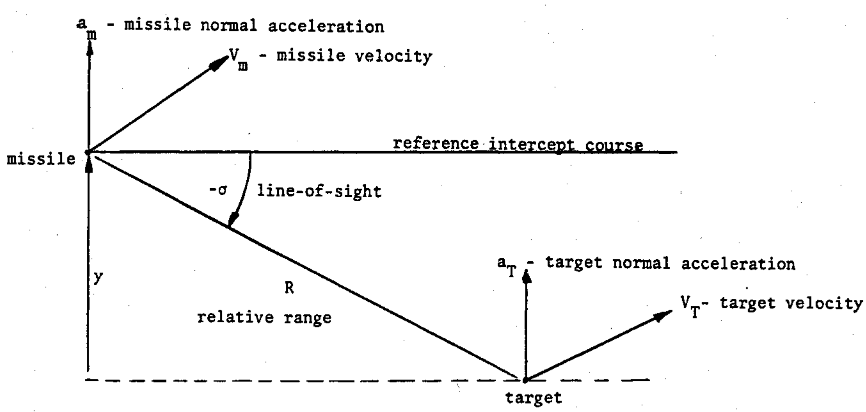

Fig. 1 Intercept geometry.

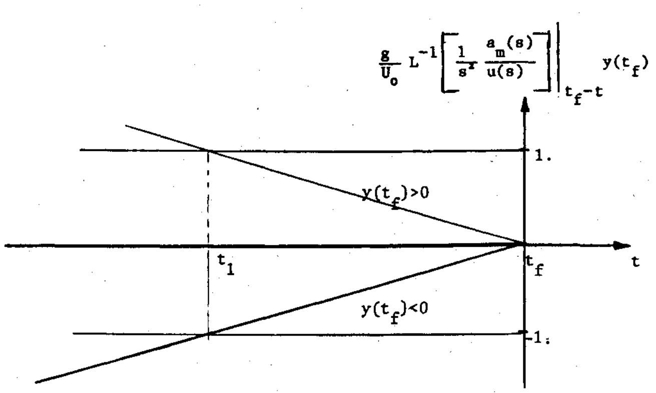

Fig. 2 Argument of the saturation function in Eqs. (12) and (13) vs time for minimum phase missile.

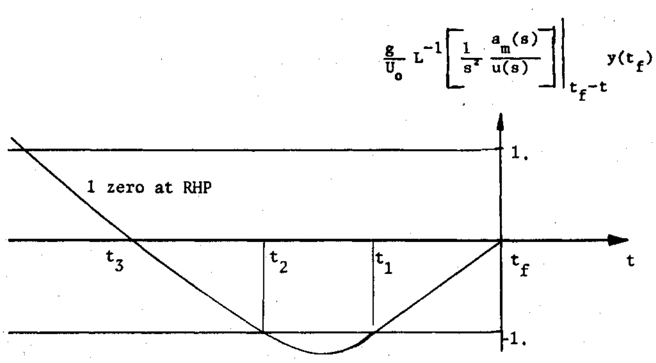

Fig. 3 Argument of the saturation function in Eqs. (12) and (13) vs time for nonminimum phase missile with one zero at RHP.

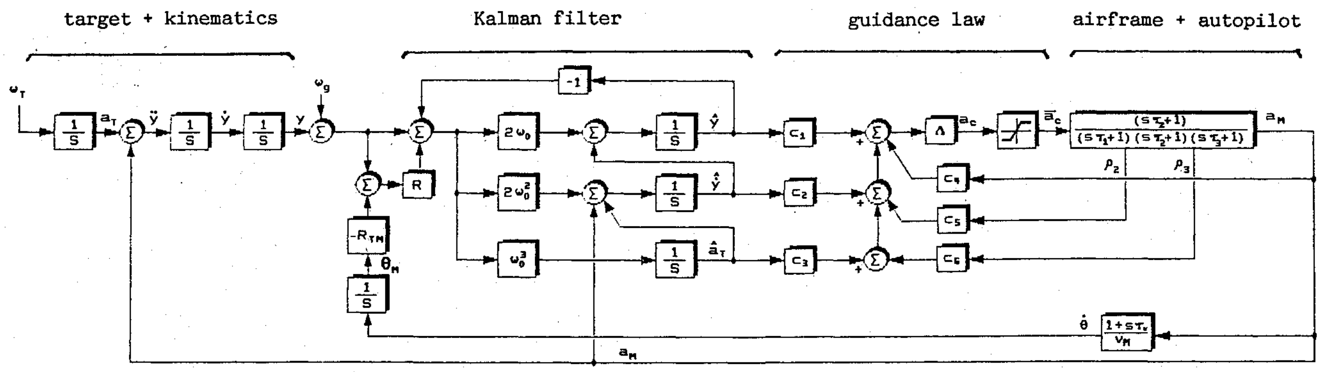

Fig. 4 Schematic diagram of the guidance loop.

(12) and (13). This saturation depends on $y(t_f)$. Consequently, for a stable minimum phase missile, the optimal guidance law is

$$u(t) = U_0 \operatorname{Sat}[(\Lambda/U_0)\Lambda(t_f - t) Z(t_f - t)] \tag{14}$$

where

$$\Lambda(\cdot) = \text{guidance gain}, \qquad Z(\cdot) = \text{zero effort miss}$$

are given by

$$\Lambda(t_f - t) = \frac{\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right)\Big|_{t_f - t}}{\frac{1}{g} + \int_0^{t_f - t}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right)\right]^2 \mathrm{d}r} \tag{15}$$

$$Z(t_f - t) = y(t) + (t_f - t)v(t) - \left[\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{a_m(0)}\right)\Big|_{t_f - t}\ \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{a_m(s)}{p_m(0)}\right)\Big|_{t_f - t}\right] \times \begin{bmatrix} a_m(t) \\ p_m(t) \end{bmatrix} \tag{16}$$

This is exactly the practical solution of Eq. (7). Equations (14-16) give the general structure of the optimal guidance law for a minimum phase arbitrary-order missile. One can see that the dominant feature, from the guidance law point of view, is the missile's ramp response.

C. Nonminimum Phase Missile

For a nonminimum phase missile, the ramp response is not monotonically increasing. (To be precise, this section applies to a more restricted class of missiles with nonmonotonous ramp response.) As an example, Fig. 3 describes the behavior of the argument of the saturation function in Eqs. (12) and (13) for a missile with one zero at RHP. For $t_2 < t < t_f$, Eqs. (14-16) are the optimal solution; however, for $t < t_2$ they are not. Close inspection of the problem will discover that the gain given by Eqs. (14-16) is smaller than the optimal, and the switching times t_1, t_2, and t_3 are given only implicitly. Since the exact optimal law is complicated to solve and implement, in the sequel only the suboptimal/practical guidance law will be considered.

IV. Example

As an example we will consider here guidance of a missile whose airframe and autopilot model is described by a third-order transfer function with one zero, i.e.,

$$H(s) = \frac{a_m(s)}{u(s)} = \frac{s\tau_z + 1}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)} \tag{17}$$

In this example it is assumed that the target's acceleration a_T is described by a step function whose initiation time is uniformly distributed over the flight time.

Figure 4 shows the schematic diagram of the guidance loop. The diagram includes the target model, where $w_T(t)$ represents the random target's maneuver[13]; the kinematics; the glint noise—$w_g(t)$; the steady-state Kalman filter, which produces the best estimates of the state variables (y, v, a_T)[11] (see Appendix B); the guidance law (c_i, $i = 1, \ldots, 6$); the saturation function; the model of the airframe and autopilot; and a model of the radome refraction.[14] The radome refraction slope is R, V_M is the missile's velocity, R_{TM} is the missile-target range, $R_{TM} = V_c(t_f - t)$, where V_c is the closing velocity, and τ_α is the turning rate time constant of the missile. We consider and compare the performance of three guidance laws: a) the full-order suboptimal guidance law; b) the first-order approximation of the guidance law; and c) the proportional navigation. The comparison is performed by computation of the root-mean-square (rms) miss due to the target's acceleration and glint noise by the SLAM analysis.[9]

A. Full-Order Suboptimal/Practical Guidance Law

In order to drive the full-order suboptimal/practical law, we use the following state space description of the missile, derived from Eq. (17)

$$\frac{d}{dt}\begin{bmatrix} a_m \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \tau_z \\ 0 & 0 & 1 \\ -\dfrac{1}{a_3} & \dfrac{\tau_z - a_1}{a_3} & -\dfrac{a_2}{a_3} \end{bmatrix} \begin{bmatrix} a_m \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{a_3} \end{bmatrix} u(t) \quad (18)$$

where $a_1 = \tau_1 + \tau_2 + \tau_3$, $a_2 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$, and $a_3 = \tau_1\tau_2\tau_3$. The full-order guidance law is given by

$$c_1 = 1$$

$$c_2 = t_f - t$$

$$c_3 = \tfrac{1}{2}(t_f - t)^2$$

$$c_4 = \mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{a_m(0)}\right]\Bigg|_{t_f-t}$$

$$c_5 = \mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{p_2(0)}\right]\Bigg|_{t_f-t}$$

$$c_6 = \mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{p_3(0)}\right]\Bigg|_{t_f-t} \quad (19)$$

$$\Lambda(t_f - t) = \frac{\mathcal{L}^{-1}[(1/s^2)H(s)]\Big|_{t_f-t}}{\dfrac{1}{g} + \displaystyle\int_0^{t_f-t}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2}H(s)\right)\right]^2 d\tau}$$

$$g = 10^{10} \quad (20)$$

The command acceleration, before the saturation, is

$$u(t) = \Lambda(t_f - t)[c_1\hat{y} + c_2\hat{v} + c_3\hat{a}_T - c_4 a_m - c_5 p_2 - c_6 p_3] \quad (21)$$

and

$$\frac{a_m(s)}{a_m(0)} = \frac{a_3 s^2 + a_2 s + a_1 - \tau_z}{a_3 s^3 + a_2 s^2 + a_1 s + 1}$$

$$\frac{a_m(s)}{p_2(0)} = \frac{a_3 s + a_2 + \tau_z(\tau_z - a_1)}{a_3 s^3 + a_2 s^2 + a_1 s + 1}$$

$$\frac{a_m(s)}{p_3(0)} = \frac{a_3(1 + s\tau_z)}{a_3 s^3 + a_2 s^2 + a_1 s + 1} \quad (22)$$

The coefficients of the target's acceleration c_3 is taken as in Ref. 11. The coefficients c_4, c_5, and c_6 are precomputed by substitution of Eq. (22) into Eq. (19), respectively.

B. First-Order Approximate Guidance Law

Here, in order to derive the guidance law, we approximate the missile transfer function by a single pole, i.e.,

$$H(s) = \frac{1}{s\tau_a + 1} \quad (23)$$

where $\tau_a = \tau_1 + \tau_2 + \tau_3 + \tau_z$. Then c_1, c_2, and c_3 are unchanged, $c_5 = 0$, $c_6 = 0$

$$c_4 = \tau_a^2[e^{-T} + T - 1], \qquad T = \frac{t_f - t}{\tau_a} \quad (24)$$

and

$$\Lambda(t_f - t) = \frac{\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\dfrac{1}{\tau_a + 1}\right]\Bigg|_{t_f-t}}{\dfrac{1}{g} + \displaystyle\int_0^{t_f-t}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{\tau_a + 1}\right)\right]^2 d\tau}, \quad g = 10^{10} \quad (25)$$

which gives [Eq. (6) in Ref. 6] for $g \to \infty$.

C. Proportional Navigation

Proportional navigation is derived if the missile is assumed to have instantaneous response, i.e., $a_m(s)/u(s) = 1$. Then we have c_1 and c_2 unchanged; $c_i = 0$, $i = 3$, 4, 5, and 6; and

$$\Lambda(t_f - t) = \frac{3}{(t_f - t)^2}, \qquad g \to \infty \quad (26)$$

D. Results

This section presents representative results of the performance of the missile model and guidance laws previously described.

Figure 5 presents curves of the effective navigation ratio, $N' = (t_f - t)^2\Lambda(t_f - t)$, vs time-to-go $t_{go} = t_f - t$ of the first-order approximate law and the third-order guidance law for minimum and nonminimum phase missiles, respectively. One can see that the effective navigation ratio goes to the positive infinity ($g \to \infty$) for the first-order approximation and third-order minimum phase case. For the third-order nonminimum phase case, the effective navigation ratio behaves more wildly and goes to the negative infinity.

The following analysis is presented for the target's acceleration and glint noise. The target's acceleration changes from 0 g to 5 g or -5 g (with equal probability). The initiation instant of the maneuver is uniformly distributed over the intercept period of $t_f = 5$ s. The glint noise has spectral density of 1 m^2/Hz.

Figure 6 shows the rms miss distance vs the value of the missile's autopilot zero for an unconstrained missile. One can see, as may be expected, the uniform performance of the third-order full-state feedback guidance law. For the minimum phase case, the difference between the first-order approximation and the third-order guidance law is minor. However, for the nonminimum phase case, one can see the superiority of the full-order guidance law.

Figures 7 and 8 present the rms miss distance vs the autopilot's zero location for a missile constraint of 50 g and 30 g,

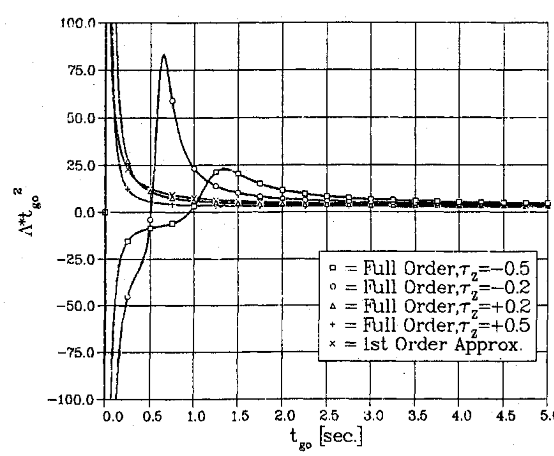

Fig. 5 Effective navigation ratio vs time-to-go for a first-order approximate guidance law and a third-order guidance law for minimum and nonminimum missile.

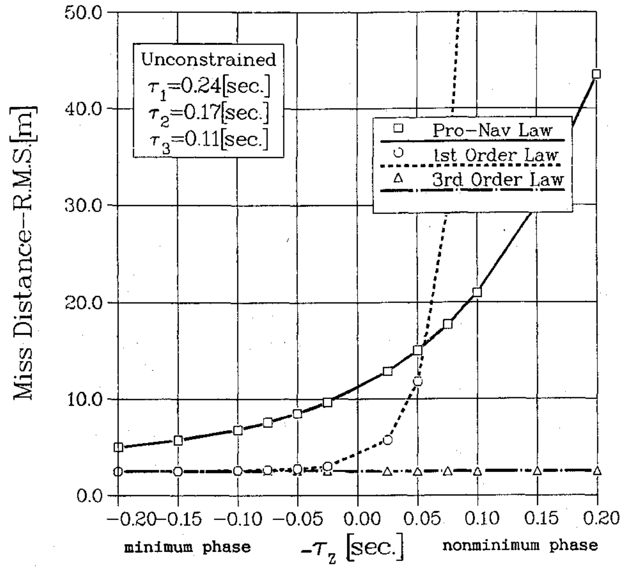

Fig. 6 The rms miss distance vs the zero location for unconstrained missile.

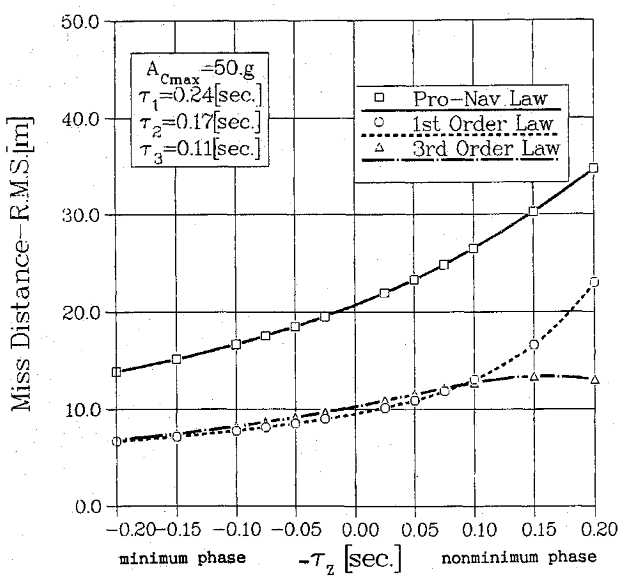

Fig. 7 The rms miss distance vs the zero location for a missile acceleration constraint of 50 *g*.

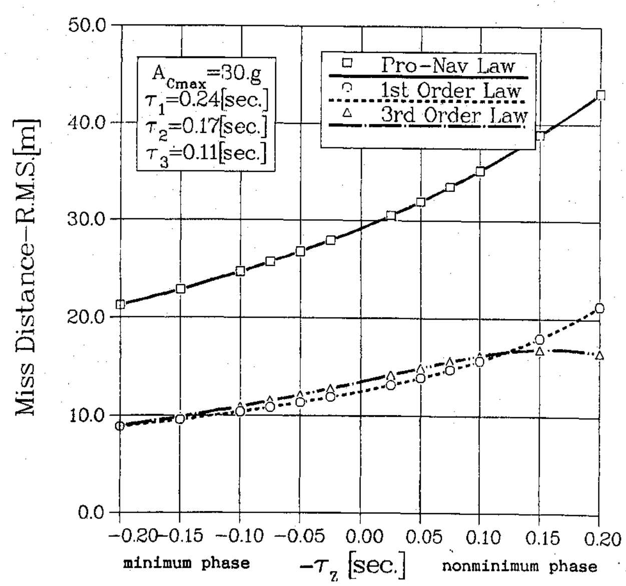

Fig. 8 The rms miss distance vs the zero location for a missile acceleration constraint of 30 *g*.

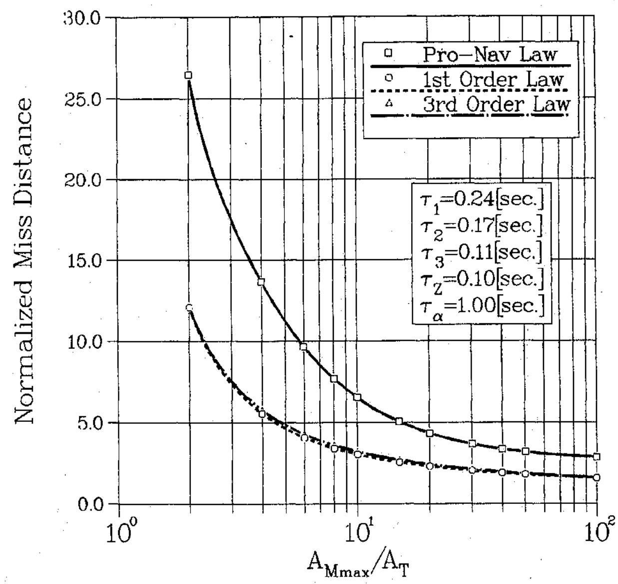

Fig. 9 Normalized miss vs the normalized missile maneuverability for a minimum phase missile.

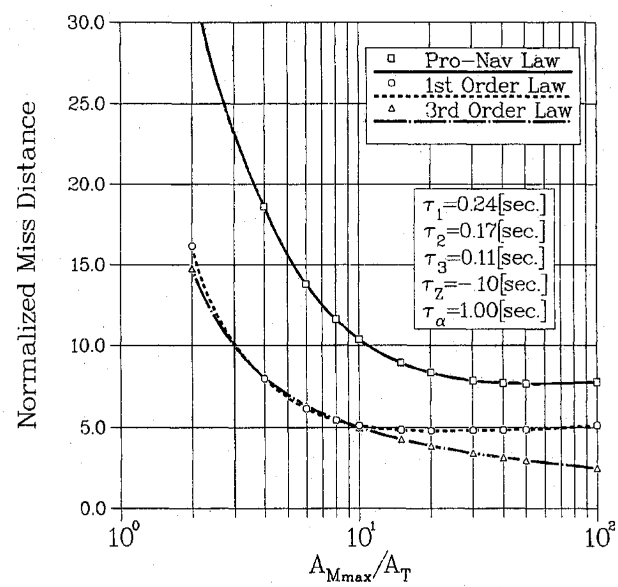

Fig. 10 Normalized miss vs the normalized missile maneuverability for a nonminimum phase missile.

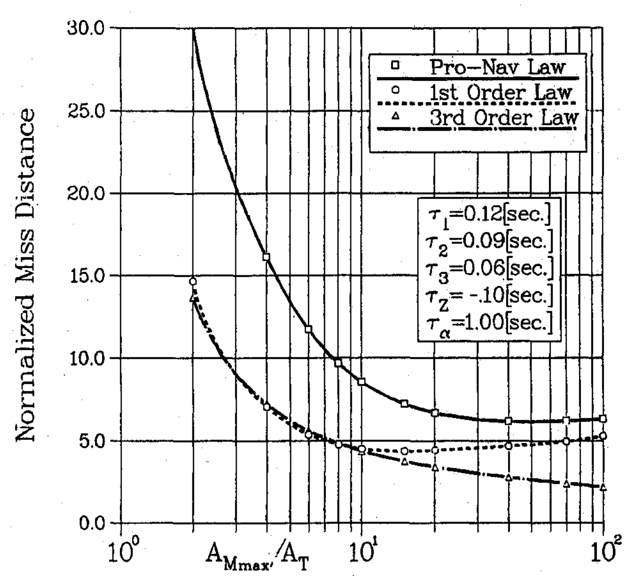

Fig. 11 Normalized miss vs the normalized missile maneuverability for a nonminimum phase missile.

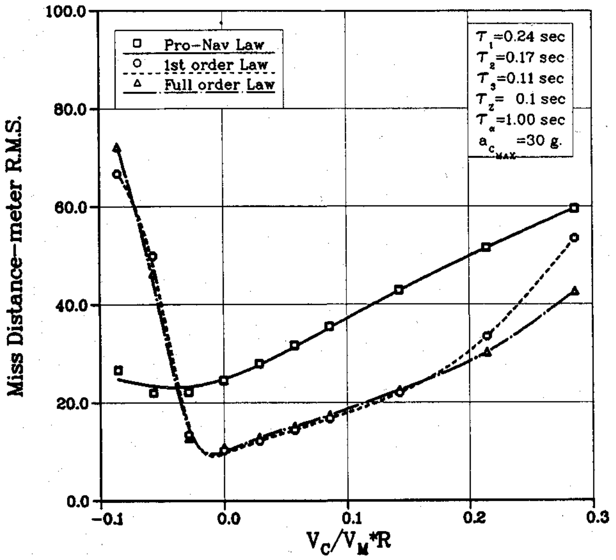

Fig. 12 Sensitivity of the rms miss to the normalized radome refraction slope for a minimum phase missile and $\tau_\alpha = 1$ s.

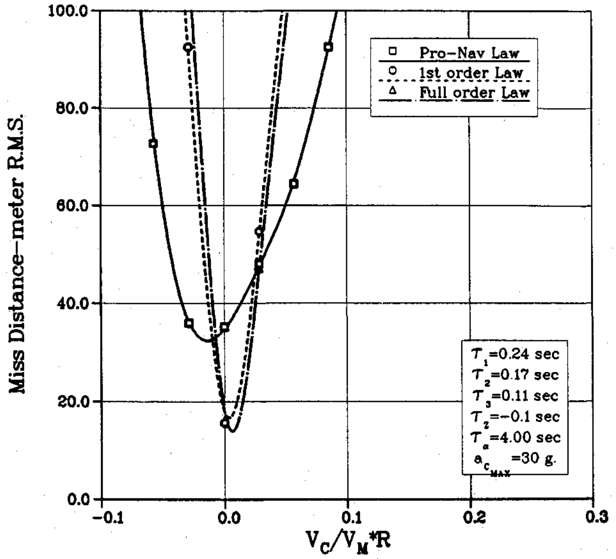

Fig. 14 Sensitivity of the rms miss to the normalized radome refraction slope for a nonminimum phase missile and $\tau_\alpha = 4$ s.

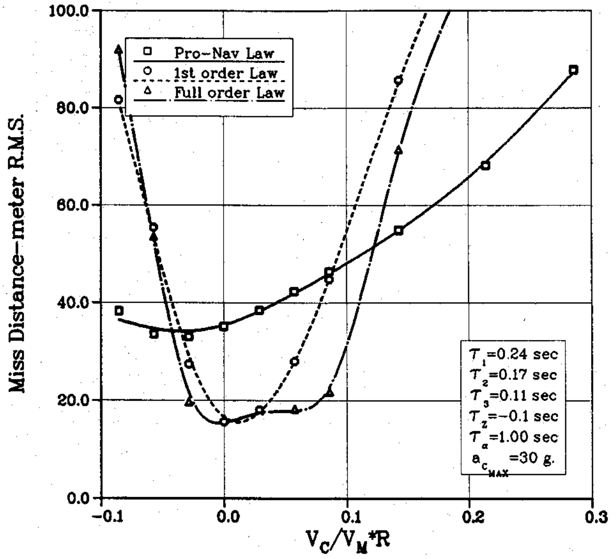

Fig. 13 Sensitivity of the rms miss to the normalized radome refraction slope for a nonminimum phase missile and $\tau_\alpha = 1$ s.

respectively. One can see that the acceleration constraint degrades the performance. The third-order guidance law with the constraint is now superior for a smaller range of the zero location.

Figures 9–11 present results for a constrained missile in a normalized form, i.e., curves of the normalized miss; rms miss/rms miss without constraint and full-order guidance law vs the normalized missile's maneuverability; maximal missile acceleration/target acceleration ($a_{m_{\max}}/a_T$ for $a_T = 5\ g$). From Fig. 9, one can see that, for a minimum phase missile, the full-order guidance law is no better than the first-order approximation. In other words, for such an airframe the first-order approximation is sufficient, and the higher-order guidance law gives negligible improvement. However, for a nonminimum phase missile, one can see, from Figs. 10 and 11, for $a_{m_{\max}}/a_T > 10$ (i.e., $a_{m_{\max}} = 50\ g$ at $a_T = 5\ g$), the gain achieved with the use of the third-order full-state feedback guidance law. By comparison of Figs. 10 and 11, one can trade off between maneuverability and response time (the missile depicted in Fig. 11 has a shorter response). The same results can be deduced from Figs. 7 and 8 as well. (The rms miss without constraint and full-order guidance law for the missile depicted in Figs. 9 and 10 is 2.56 m and in Fig. 11 is 2.36 m.) The presented comparison between the full-order guidance law and the first-order approximate guidance law may be viewed as well as a sensitivity/robustness check with respect to the gains c_4, c_5, and c_6 and the associated parameters τ_1, τ_2, τ_3, and τ_z. Namely, for a minimum phase missile, the sensitivity is low. For a nonminimum phase missile, there is a certain amount of sensitivity which depends also on the level of the acceleration constraint.

Figures 12–14 present sensitivity/robustness studies with respect to the radome refraction slope for a missile acceleration constraint of 30 g. The figures present the rms miss distance vs the closing-velocity-to-missile-velocity ratio times the radome refraction slope[14] for different values of the missile turning rate time constant. Figure 12 is for a minimum phase airframe. One can see that the difference between the first- and third-order guidance law is practically negligible. For a nonminimum phase airframe in Figs. 13 and 14, for $\tau_\alpha = 1$, and 4 s, respectively, one can see the improved robustness with respect to the radome refraction slope of the third-order guidance law.

V. Conclusions

The general form for optimal guidance law enables one to systematically synthesize modern guidance laws for high-order and acceleration constrained missiles. For a minimum phase missile, the optimal guidance law is the guidance law for an unconstrained missile with saturation on the commanded acceleration. For a nonminimum phase missile, this is only a suboptimal guidance law, and the optimal controller is more complicated. Comparison of the proportional navigation, first-order approximation and full-order guidance laws is performed on a third-order minimum and nonminimum phase model of a missile. The robustness of these guidance laws is subject to an analysis. The main conclusions are that for a minimum phase missile the full-order guidance law does not give improved performance over the first-order approximation, whereas for a nonminimum phase missile there are situations (combinations of poles and zeros location and acceleration constraint) when the full-order guidance law is worth use.

Appendix A: Equivalence of Quadratic Performance Indices

Let us consider the following two optimization problems on a finite interval.

Problem I.

$$J_1 = \frac{1}{2} x_f^T P_f x_f + \frac{1}{2} \int_t^{t_f} \left[x^T Q x + 2u^T S x + u^T R u \right] d\tau$$

$\min J_1$, subject to

$$\dot{x} = Ax + Bu$$

Problem II.

$$J_2 = \frac{1}{2} x_f^T W_f x_f + \frac{1}{2} \int_t^{t_f} u^T V u \, d\tau$$

$\min J_2$, subject to

$$\dot{x} = Fx + Hu$$

Lemma. Optimization problems I and II are equivalent if

$$F = A - BR^{-1}S - BR^{-1}B^T\bar{P}$$

$$H = B, \qquad W_f = P_f - \bar{P}, \qquad V = R$$

where $\bar{P}$ is the solution of the algebraic matrix Riccati equation

$$(A + BR^{-1}S)^T\bar{P} + \bar{P}(A - BR^{-1}S) - \bar{P}B^TR^{-1}B\bar{P} + Q - S^TR^{-1}S = 0$$

Proof.

a) Problem I is solved then

$$u = R^{-1}(S + B^TP)x$$

$$-\dot{P} = (A - BR^{-1}S)^TP + P(A - BR^{-1}S) - PB^TR^{-1}BP + Q - S^TR^{-1}S$$

$$P(t_f) = P_f, \; t \le t_f$$

the solution is then

$$P(t) = \bar{P} + \Phi^T(t_f,t)(P_f - \bar{P}) \times [I + \int_t^{t_f} \Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)(P_f - P)\, d\tau]^{-1}\Phi(t_f,t) \qquad t \le t_r$$

$$\frac{d}{dt}\Phi(t,t_0) = (A - BR^{-1}S - B^TR^{-1}B\bar{P})\Phi(t,t_0)\Phi(t_0,t_0) = I$$

and

$$(A - BR^{-1}S)^T\bar{P} + \bar{P}(A - BR^{-1}S) - \bar{P}B^TR^{-1}B\bar{P} + Q - S^TR^{-1}S = 0$$

Then the closed loop is

$$\dot{x} = \{A - BR^{-1}S - BR^{-1}B^T\bar{P} - BR^{-1}B\Phi^T(t_f,t) \times (P_f - \bar{P})[I + \int_t^{t_f} \Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)(P_f - \bar{P})\, d\tau]^{-1} \times \Phi(t_f,t)\}x$$

b) The solution of Problem II is

$$u = -V^{-1}H^TWx$$

$$-\dot{W} = F^TW + WF - WH^TV^{-1}HW$$

$$W(t_f) = W_f, \qquad t \le t_f$$

whose solution is

$$W(t) = \psi^T(t_f,t)W_f[I + \int_t^{t_f} \psi(t_f,\tau)HV^{-1}H^T\psi^T(t_f,\tau)W_f\, d\tau]^{-1} \times \psi(t_f,t)$$

$$\frac{d}{dt}\psi(t,t_f) = F\psi(t,t_0); \qquad \psi(t_0,t_0) = I$$

and

$$\dot{x} = \{F - HV^{-1}h\psi^T(t_f,t)W_f\left[I + \int_t^{t_f} \psi(t_f,\tau)HV^{-1}H^T\psi^T \times (t_f,\tau)W_f\, d\tau\right]^{-1}\psi(t_f,t)\}x$$

and the equivalance directly follows.

Appendix B: General Problem and Solution

The problem considered here is to minimize the quadratic performance index of Eq. (1) subject to the constraints of Eqs. (2) and (3).

The solution is obtained by the minimization of the Hamiltonian[10,12]

$$H(x,p,u) = \tfrac{1}{2}u^TRu + p^T[Ax + Bu] \tag{B1}$$

with the constraint $u^Tu < U_0^2$ where P is the costate vector.

1) First let us assume that $u^Tu < U_0^2$ so that H is derivable. Then the solution is derived from

$$H_u = 0 \tag{B2a}$$

$$-H_x^T = \dot{p} \tag{B2b}$$

$$p(t_f) = \frac{\partial}{\partial x(t_f)}\left[\frac{1}{2}x^T(t_f)G\,x(t_f)\right] = G\,x(t_f) \tag{B2c}$$

then from Eq. (B2a) the feedback is

$$u = -R^{-1}B^Tp \tag{B3}$$

and we have the following two-point boundary value problem

$$\dot{x} = Ax - BR^{-1}B^Tp; \qquad x(t_0) = x_0 \tag{B4a}$$

$$\dot{p} = -A^Tp; \qquad p(t_f) = G\,x(t_f) \tag{B4b}$$

From Eqs. (B4a) and (B3), we have

$$x(t_f) = \Phi(t_f,t)x(t) - \int_t^{t_f} \Phi(t_f,\tau)BR^{-1}B^Tp(\tau)\, d\tau \tag{B5}$$

where

$$\dot{\Phi}(t,t_0) = A\,\Phi(t,t_f); \qquad \Phi(t_0,t_0) = I \tag{B6}$$

and from Eqs. (B4b) and (B2c)

$$p(t) = \Phi^T(t_f,t)\,G\,x(t_f) \tag{B7}$$

Next, substitution of Eq. (B7) into Eq. (B5) and rearrangement give

$$\left[I + \int_t^{t_f} \Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)G\, d\tau\right]x(t_f) = \Phi(t_f,t)x(t) \tag{B8}$$

and finally Eqs. (B8), (B7), and (B3) give

$$u(t) = -R^{-1}B^T\Phi^T(t_f,t)G\left[I + \int_t^{t_f}\Phi(t_f,\tau)\right.$$

$$\left.\times BR^{-1}B^T\Phi^T(t_f,\tau)G\,\mathrm{d}\tau\right]^{-1}\Phi(t_f,t)x(t) \qquad \text{(B9)}$$

2) Now, let us assume that u reached the constraint of Eq. (3) so that the Hamiltonian of Eq. (B1) is underivable and one should look for a solution by direct minimization of H according to Pontriagyn's minimum principle,[12] i.e.,

$$\min H(x, p, u)$$

$$u^T u \leq U_0^2 \qquad \text{(B10)}$$

and it is necessary that

$$H(x^*, p^*, u^*) \leq H(x^*, p^*, u) \qquad \text{(B11)}$$

where ( )* denotes the values at optimum.

So we find that

$$\tfrac{1}{2}u^{*T}Ru^* + p^{*T}[Ax^* + Bu^*] \leq \tfrac{1}{2}u^T Ru$$

$$+ p^{*T}[Ax^* + Bu] \qquad \text{(B12)}$$

and we have

$$u^{*T}[B^T p^* + \tfrac{1}{2}Ru^*] \leq u^T[B^T p^* + \tfrac{1}{2}Ru] \qquad \text{(B13)}$$

The optimal control must minimize the scalar product

$$< u,\ B^T p^* + \tfrac{1}{2}Ru > \qquad \text{(B14)}$$

i.e., u and $B^T p^* + \tfrac{1}{2}Ru$ should be parallel and in opposite direction. Figure B1 shows a geometric interpretation of Eq. (B14).

Since $u^{*T}u^* = U_0^2$ we claim that the optimal feedback is

$$u^* = -U_0\frac{R^{-1}B^T p^*}{\|R^{-1}B^T p^*\|} \qquad \text{(B15)}$$

Since the Hamiltonian is unconstrained with respect to $x(t_f)$ and p, Eqs. (B2b) and (B2c) must be satisfied, i.e., we have the two-point boundary value problem

$$\dot{x} = Ax + Bu; \qquad x(t_0) = x_0 \qquad \text{(B16a)}$$

$$p = -A^T p; \qquad p(t_f) = Gx(t_f) \qquad \text{(B16b)}$$

where u is taken from Eq. (B15).

3) Finally, from the previous section, we deduce that the solution to the problem considered here is the two-point boundary value problem

$$\dot{x} = Ax + Bu; \qquad x(t_0) = x_0 \qquad \text{(B17a)}$$

$$\dot{p} = -A^T p; \qquad p(t_f) = Gx(t_f) \qquad \text{(B17b)}$$

$$u = -U_0\,\mathrm{Sat}[(1/U_0)R^{-1}B^T p] \qquad \text{(B17c)}$$

and from Eq. (B17b) we have

$$p(t) = \Phi^T(t_f,t)\,Gx(t_f) \qquad \text{(B18)}$$

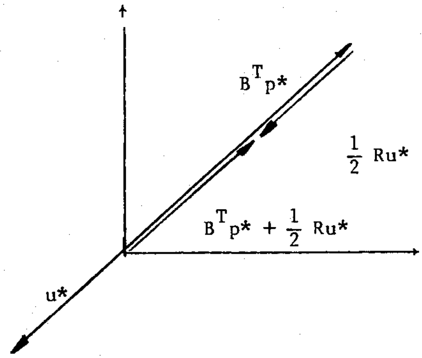

Fig. B1 Geometric interpretation of Eq. (B14).

and from Eq. (B17a) we have the implicit equation for $x(t_f)$

$$x(t_f) = \Phi(t_f,t)x(t) - \int_t^{t_f}\Phi(t_f,\tau)BU_0\,\mathrm{Sat}[(1/U_0)R^{-1}B^T\Phi^T(t_f,\tau)$$

$$\times Gx(t_f)]\,\mathrm{d}\tau \qquad \text{(B19)}$$

and

$$u(t) = -U_0\,\mathrm{Sat}[(1/U_0)R^{-1}B^T\Phi^T(t_f,t)Gx(t_f)] \qquad \text{(B20)}$$

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